MATH 3060: HW 8 Solution

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(1) Show that the set $\{ \frac{k}{2^\ell} \in \mathbb{R} : k, \ell \in \mathbb{Z} \}$ is of 1st category, but not nowhere dense.

Sol) Let $E \subseteq (\mathbb{R}, |\cdot|)$ be the given set as a metric subspace.

ⓐ E is of first category:

Note that E is a countable subset and $(\mathbb{R}, |\cdot|)$ has no isolated points.

Therefore, by Prop. 4.8(c), E is of first category.

ⓑ E is not nowhere dense:

Note that $\overline{E} = \mathbb{R}$ (i.e. every real number has a (unique) binary representation.)

(For some details, see [Bartle: §2.5 under subsection: Binary Representation])

Therefore, $(\overline{E})^\circ = \mathbb{R} \neq \phi$. Hence, E is not nowhere dense.

(2) Show that $\mathcal{E} = \{ f \in C[0,1] : \int_0^1 f(x)dx \neq 0 \}$ is a residual set in $(C[0,1], d_\infty)$.

Sol) Let $\Sigma = C[0,1] \setminus \mathcal{E} = \{ f \in C[0,1] : \int_0^1 f(x) = 0 \}$.

It suffices to show that Σ is of first category.

Actually, Σ is nowhere dense in $(C[0,1], d_\infty)$ (hence is of first category):

We first show that Σ is closed in $(C[0,1], d_\infty)$:

Given $(f_n)_{n=1}^{\infty} \subseteq \Sigma$ with $\lim_{n\to\infty} f_n = f \in C[0,1]$, showing $f \in \Sigma$:

Given $\varepsilon > 0$, since $\lim_{n\to\infty} f_n = f$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$,

$\|f_n - f\|_\infty < \varepsilon$. Therefore, $\left| \int_0^1 f(x)dx \right| = \left| \int_0^1 (f(x) - f_n(x))dx \right|$ (since $\int_0^1 f_n(x)dx = 0$)

$\leq \int_0^1 \|f - f_n\|_\infty dx < \varepsilon$ for any $\varepsilon > 0$. Therefore, $\int_0^1 f(x)dx = 0$, i.e. $f \in \Sigma$.

We then show that Σ is nowhere dense by showing $\Sigma^\circ = \phi$:

Given any $f \in \Sigma$ and $\varepsilon > 0$, define $g(x) = f(x) + \frac{\varepsilon}{2} \in C[0,1]$.

Then $d_\infty(f, g) = \frac{\varepsilon}{2} < \varepsilon$. $\therefore g \in B_\varepsilon(f)$;

Meanwhile, $\int_0^1 g(x)dx = \int_0^1 (f(x) + \frac{\varepsilon}{2})dx = \frac{\varepsilon}{2} \neq 0$. $\therefore g \notin \Sigma$.

Therefore, $B_\varepsilon(f) \nsubseteq \Sigma$, hence $\Sigma^\circ = \phi$.

(3) Show that $\mathcal{P} = \{ p \in C[0,1] : p \text{ is a polynomial} \}$ is a set of 1^{st} category.

Sol) Write $\mathbb{P} = \bigcup_{n=1}^{\infty} \mathbb{P}_n$, where $\mathbb{P}_n = \{ P \in \mathbb{P} \mid \deg P \leq n \}$.

It suffices to show that for any $n \in \mathbb{N}$, $\mathbb{P}_n$ is nowhere dense in $(C[0,1], d_\infty)$:

We first show that $\mathbb{P}_n$ is closed in $(C[0,1], d_\infty)$:

Given $(P_m)_{m=1}^{\infty} \subseteq \mathbb{P}_n$ with $\lim_{m \to \infty} P_m = f \in C[0,1]$, showing $f \in \mathbb{P}_n$:

Since f is the uniform limit of smooth functions P_m, f is smooth on $[0,1]$,

hence admitting Taylor series expansion at $\frac{1}{2}$: $f(x) = \sum_{k=0}^{\infty} a_k (x - \frac{1}{2})^k$ on $[0,1]$.

Since for any $m \in \mathbb{N}$, $\deg P_m \leq n$, therefore for $k > n$, $a_k = \frac{f^{(k)}(\frac{1}{2})}{k!} = \lim_{m \to \infty} \frac{P_m^{(k)}(\frac{1}{2})}{k!} = 0$.

$\therefore$ $f(x) = \sum_{k=0}^{n} a_k (x - \frac{1}{2})^k$ is a polynomial on $[0,1]$ with $\deg(f) \leq n$. Hence, $f \in \mathbb{P}_n$.

We then show that $\mathbb{P}_n$ is nowhere dense by showing $(\mathbb{P}_n)^\circ = \phi$:

Given any $P \in \mathbb{P}_n$ and $\varepsilon > 0$, consider $g(x) := P(x) + \frac{\varepsilon}{2} \cdot e^{-x} \in C[0,1]$.

Then $d_\infty(g, P) = \frac{\varepsilon}{2} < \varepsilon$, hence $g \in B_\varepsilon(P)$;

Meanwhile, $g \notin \mathbb{P}_n$ as e^{-x} is not a polynomial on $[0,1]$.

Therefore, $B_\varepsilon(P) \not\subseteq \mathbb{P}_n$, hence $(\mathbb{P}_n)^\circ = \phi$.

(4) Show that a countable metric space with no isolated point cannot be complete.

Sol) Suppose on the contrary, (X, d) is a complete countable metric space with no isolated points. Since X is countable, $X = \bigcup_{n=1}^{\infty} \{x_n\}$, where $x_n \in X$.

Since (X,d) has no isolated points, by Prop. 4.7 (c), $\{x_n\}$ is nowhere dense.

Therefore, $X = \bigcup_{n=1}^{\infty} \{x_n\}$ is of first category.

Since (X,d) is complete, by Baire Category Theorem, X has empty interior.

This is a contradiction as $X^\circ = X \neq \phi$.

Therefore, any countable metric space with no isolated points cannot be complete.

(5) Let $l_2 = \{ \{x_n\}_{n=1}^{\infty} : \sum_{n=1}^{\infty} x_n^2 < \infty \}$ with metric

$$d_2(\{x_n\}, \{y_n\}) = \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}.$$

Show that $H = \{ \{x_n\}_{n=1}^{\infty} \in l_2 : |x_n| \le \frac{1}{n} \}$ is nowhere dense in (l_2, d_2).

Sol) Recall that $H \subseteq (l_2, d_2)$ is closed by HW4, Q3.

$\therefore$ To show H is nowhere dense, it suffices to show that $H^\circ = \phi$.

Given any $(x_n)_{n=1}^{\infty} \in H$ and $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$.

Define $(y_n)_{n=1}^{\infty}$ by $y_n = \begin{cases} x_n, & n \neq 2N. \\ x_{2N} + \frac{2}{N}, & n = 2N. \end{cases}$

then $\sum_{n=1}^{\infty} |y_n|^2 = \left(\sum_{\substack{n=1 \\ n \neq 2N}}^{\infty} |x_n|^2 \right) + \left| x_{2N} + \frac{2}{N} \right|^2 < +\infty$. Hence $(y_n) \in l_2$.

Also, $d_2((x_n), (y_n)) = \left(\sum_{\substack{n=1 \\ n \neq 2N}}^{\infty} |0|^2 + \left|\frac{2}{N}\right|^2 \right)^{\frac{1}{2}} = \frac{2}{N} < \varepsilon$. Therefore, $(y_n) \in B_\varepsilon((x_n))$.

Meanwhile, $|y_{2N}| = \left| x_{2N} + \frac{2}{N} \right| > \frac{1}{N} - |x_{2N}| \geq \frac{1}{N} - \frac{1}{2N} = \frac{1}{2N}$. $\therefore (y_n) \notin H$.

Therefore, $B_\varepsilon((x_n)) \nsubseteq H$ for any $(x_n) \in H$, $\varepsilon > 0$. Hence, $H^\circ = \phi$.

As a result, H is nowhere dense.